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Sensitivity to Initial Conditions in Self-Organized Critical Systems

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We discuss sensitivity to initial conditions in a model for avalanches in granular media displaying self-organized criticality. We show that damage, due to a small perturbation in initial conditions, does not spread. The damage persists in a statistically time-invariant and scale-free form. We argue that the origin of this behavior is the Abelian nature of the model, which generalizes our results to all models with Abelian properties, including the BTW model and the Manna model. An ensemble average of the damage leads to seemingly time dependent damage spreading. Scaling arguments show that this numerical result is due to the time lag before avalanches reach the initial perturbation.

KEY WORDS: Self-organized criticality; Oslo model; abelian sandpile; sensitivity to initial conditions.

There is evidence that the dynamics of a pile of rice may display self-organized criticality (SOC).[(1)] In careful experiments where elongated rice grains were slowly dropped between two glass plates, Frette *et al.* found scale-free behavior in a rice pile.[(2)] In a slowly driven pile, the angle of repose evolves to a stationary state and the behavior of the system is dominated by a scale-free avalanche size density and punctuated equilibrium. This punctuation causes SOC systems to be highly non-linear, a single grain added at one end can result in an avalanche propagating through the entire system. However, SOC models typically have stick-slip dynamics[(3)] and it has yet to be established whether they allow the non-linearity to manifest itself in the form of sensitivity to initial conditions, as it does in chaotic systems.

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We have studied damage spreading in a simple one-dimensional granular model known as the Oslo model, which exhibits SOC.[4] It describes a number of slowly driven granular systems and belongs to the same universality class as a model for interface depinning in a random medium and the Burridge–Knopoff train model for earthquakes.[5–7] The Oslo model has largely resisted efforts for an analytic solution, the few exceptions have been the exact enumeration of the number of recurrent configurations,[8] the mapping of the model to the quenched Edwards–Wilkinson equation in the continuous limit,[5,9] and the transition matrix results [10] and operator algebra recently developed for the Oslo model.[11] In this letter, we add to its analytical description by illustrating its Abelian properties.[12]

We find that damage due to a small perturbation does not spread in the Oslo model as the damage is unable to evolve. It is possible to represent the perturbation in terms of commutative operators which leads to a statistically time-invariant and scale-free damage. This phenomenon is in contrast to what is seen in chaotic and equilibrium systems. We also address the results obtained by a previous study on an ensemble average of the damage, which seems to contradict our findings.[13] In fact, we show that these results are consistent with ours and that the observed behavior may be derived using simple scaling arguments.

The model: The Oslo model is defined on a one dimensional discrete lattice with L sites at positions $x = 1, 2, \dots, L$ (see Fig. 1). On the left-hand side, the boundary is a vertical wall, and on the right-hand side, the boundary is open. The height of grains at site x and time t is denoted $h(x,t)$, the local slope is defined as $z(x,t) = h(x,t) - h(x+1,t)$. Each site has a critical slope, $z_c(x)$, which takes the values 1 or 2 with equal

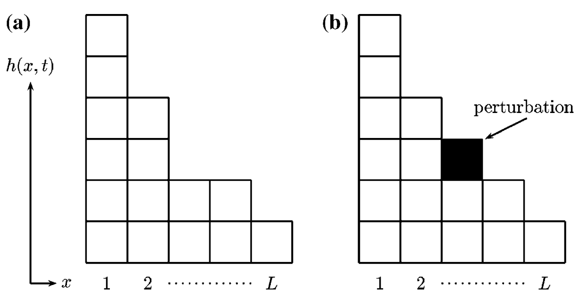

Fig. 1. (a) The Oslo model of a one-dimensional granular pile. Grains are added at the site $x = 1$ next to the vertical wall by letting $h(1, t+1) = h(1,t) + 1$. A grain at site x topples to site $x+1$ if the local slope $z(x,t) = h(x,t) - h(x+1,t)$ exceeds its critical slope $z_c(x)$. When $z(L,t) > z_c(L)$, a grain leaves the system at the open boundary, $h(L,t) \to h(L,t) - 1$. (b) The solid grain is the additional grain in the copy at time t_0.

probability. At each time step a single grain is added to the site at $x=1$. If the local slope at any site, x, exceeds its critical slope, $z(x,t) > z_c(x)$, an avalanche is initiated. The site will relax and a grain will topple from site x to site $x+1$, i.e., $h(x,t) \to h(x,t)-1$ and $h(x+1,t) \to h(x+1,t)+1$. Each time a site relaxes its critical slope is redetermined, chosen randomly from the values 1 or 2. This toppling may cause sites $x \pm 1$ to exceed their critical slopes, in which case these sites relax in turn. The avalanche will continue until the system reaches a stable configuration, when $z(x,t) \leqslant z_c(x)$ for all x.

In order to study damage spreading in the model we consider a system which has been evolved to the critical state and make an exact copy at some time, t_0. We define $h^{\text{o}}(x,t)$ and $h^{\text{c}}(x,t)$ as the heights of the original and copy, respectively, and $z_c^{\text{o}}(x)$ and $z_c^{\text{c}}(x)$ as the critical slopes of the original and copy, such that

$$\begin{array}{rcl} h^{\text{c}}(x,t_0) &=& h^{\text{o}}(x,t_0), \\ z_c^{\text{c}}(x) &=& z_c^{\text{o}}(x), \end{array} \qquad 1 \leqslant x \leqslant L. \tag{1}$$

We then perturb the copy by adding a single grain to a site, i, such that $h^{\text{c}}(i,t_0) = h^{\text{o}}(i,t_0)+1$ (see Fig. 1(b)). This extra grain is not allowed to topple until it is toppled upon by an avalanche from above. The two systems are then evolved using exactly the same sequence of random thresholds $\{z_c(x)\}$ for corresponding sites in the original and copy. We achieve this by keeping a list $\{z_c(x)\}$ of thresholds for each site and when a site relaxes its new threshold is taken as the next value in the list. Motivation for this is to consider the mapping of the model to interface depinning as it is clear that the medium the interface moves through does not change as a result of the perturbation.

We measure the damage, defined as

$$H(t,L) = \sum_{x=1}^{L} |h^{\text{o}}(x,t) - h^{\text{c}}(x,t)|. \tag{2}$$

Fig. 2 is a plot of damage versus time for a typical simulation. There appears to be no sense of temporal evolution in the data, the damage is continually fluctuating between high and low values, often returning to $H=1$ where the original and copy only differ by a single grain. The triangles in Fig. 2 indicate which of these $H=1$ configurations are repaired configurations where the extra grain in the copy is at the site which was originally perturbed and corresponding sites in the original and copy have relaxed exactly the same number of times. In repaired configurations, the

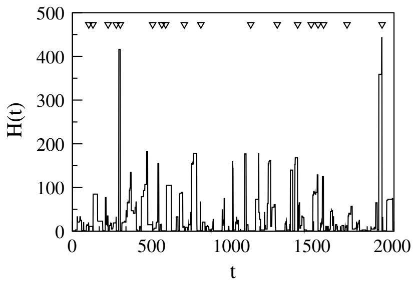

Fig. 2. Damage as a function of time for a single run, $L = 128$, where the perturbed site is $i = 16$ and we have taken $t_0 = 0$. Notice that there is no temporal evolution in the data, the damage is continually fluctuating between low and high values and frequently returns to a $H = 1$ configuration. The triangles indicate where the $H = 1$ configuration corresponds to a repaired configuration, where corresponding sites in the original and copy have relaxed an equal number of times.

difference between the two systems is *exactly equal* to the initial perturbation at t_0. The occurrence of a repaired configuration corresponds to 'resetting the clock', meaning that the damage may not evolve in a single pair of systems, it is statistically time invariant.

This behavior emerges because the Oslo model has an Abelian nature, where the commutative operation is adding one unit of *slope* to a site and allowing the system to relax. The Abelian nature follows from the fact that the sequences of critical slopes are no longer random noise, but an intrinsic property of the system where the values, although generated randomly, are treated as given *a priori*. Hence, changing the order of the relaxations will not change the sequences of thresholds. Note that the operators do not form an Abelian group as there is no inverse operation. This distinguishes the model from other so-called 'Abelian' models such as the Abelian sandpile.[12]

We introduce the notation $C(t)$ to represent the stable configuration of a system at time t. The relevant operators are the perturbation operator P_i, which simply adds one unit of slope to site i and $\hat{P}_i$, which adds one unit of slope to site i *and* allows the system to relax if necessary. Thus, $\hat{P}_i$ represents a mapping within the configuration space of $C(t)$. The evolution of $C(t)$ can be expressed by an evolution operator $\hat{T} \equiv \hat{P}_1$, that is

$$C(t+1) = \hat{T} C(t). \tag{3}$$

The operators $\hat{P}_i$ and $\hat{T}$ obey the commutation relation

$$\hat{P}_i \hat{T} C(t) = \hat{T} \hat{P}_i C(t), \tag{4}$$

which we prove elsewhere. Hence, adding slope to a system and then evolving it leads to the same configuration as evolving it and then adding slope, as we may always move the $\hat{P}_i$ operator on the right-hand side of Eq. (4) to the left-hand side of the $\hat{T}$ operator.

The perturbation we studied in the simulations was the addition of a single grain and not slope. Adding a single grain to site i will change the slopes such that $z^{c}(i,t_0)=z^{o}(i,t_0)+1$ and $z^{c}(i-1,t_0)=z^{o}(i-1,t_0)-1$. To analyze this situation, we start with a master system $C^{m}(0)$ and derive from this two other systems $C^{o}(0)$ and $C^{c}(0)$ through the relations

$$\begin{aligned} C^{o}(0) &= P_{i-1}C^{m}(0),\\ C^{c}(0) &= P_{i}C^{m}(0). \end{aligned} \tag{5}$$

It is straightforward to verify that the configurations $C^{o}(0)$ and $C^{c}(0)$ differ by one grain at site i, thus reproducing the original and perturbed systems of our simulations with the perturbative grain placed at site i (see Fig. 3). After an avalanche has reached site i, the configuration of the master is related to those of the original and copy by the operators $\hat{P}_{i-1}$ and $\hat{P}_i$, respectively, due to Eq. (4). However, the operator $\hat{P}_i$ does not have a unique inverse so there is no direct path between the original and copy.

It is clear that damage cannot *spread.* Spreading implies that the small perturbation at t_0 leads to a small amount of damage at t_0+1, which will grow until the damage saturates the system, which is the case in

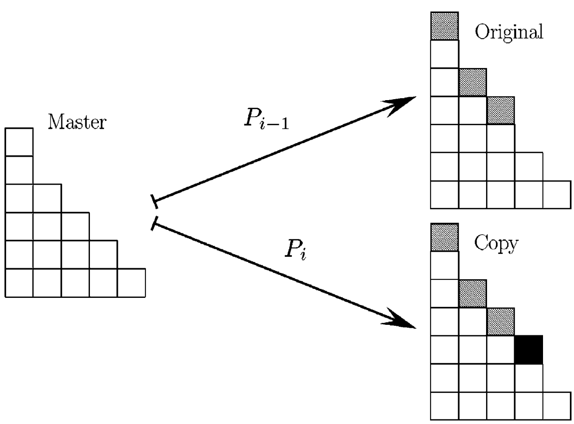

Fig. 3. The relationship between the master, $C^{m}(0)$, original, $C^{o}(0)$, and copy, $C^{c}(0)$ configurations at time $t_0=0$ (see Eq. (5)). The shaded grains are those added to the master configuration to produce configuration of the original system. The solid grain is the additional grain added to the configuration of the copy at time t_0.

a chaotic system. The Abelian nature means that the value of the damage at any time is independent of when the perturbation took place. Also, the damage does not remain constant or decay, as in an equilibrium system, because the damage is exactly that due to the avalanches which would be caused by adding slope to different sites in the master system at that time. This leads to a damage size density that is related to the avalanche size density, as easily recognized by considering the special case of perturbing site $i=1$. As the avalanche size density is scale free, we find that the damage size density is scale free also. In an infinite system the damage may become arbitrarily large, yet it will frequently return to an $H=1$ configuration! In other words such a system may be considered as lying on the edge of chaos.[14,15]

The damage in a single pair of systems is statistically time invariant, yet a previous study has found that the ensemble averaged data is not.[13] The ensemble average of damage over N runs, gives the average damage as a function of time

$$\langle H\rangle(t,L)=\frac{1}{N}\sum_{j=1}^{N}H_j(t,L,i_j), \tag{6}$$

where $H_j(t,L,i_j)$ is the damage from a single run and the variable i_j is the site of perturbation for the jth run. In ref. 13 it was found that for $i_j=L/2\ \forall j$

$$\langle H\rangle(t,L)=t^z\ G\left(\frac{t}{L^\beta}\right), \tag{7}$$

where z and β are exponents to be determined and $G(x)$ is constant for $x\ll 1$ and proportional to x^{-z} for $x\gg 1$. The apparent time dependence for time $t\ll L^\beta$ arises from the fact that the perturbed site is not allowed to relax until its neighbour relaxes. This forces all the systems into a repaired configuration at the start of the simulation and the average damage increases over time as more systems have avalanches which reach the perturbed site.

In the case where the perturbative grain was placed on a random site for each system in the ensemble, scaling arguments may be used to derive the temporal evolution of $\langle H\rangle(t,L)$. The derived equation agrees well with the simulation data. First, we calculate how long avalanches take to reach the perturbed site, which we denote as site i. The linear avalanche size, l, is known to be related to the avalanche size, s, by $s\propto l^D$, where D is the avalanche fractal dimension, $D \approx 2.25$.[5] Hence, the probability $P_l(l, L)dl$ of having an avalanche with linear length in the range $l \to l + dl$, obeys $P_l(l, L)\ dl = P_s(s, L)\ ds$, where $P_s(s, L)ds$ is the probability of having an avalanche of size s in the range $s \to s + ds$, given by the scaling ansatz

$$P_s(s, L)\ ds = s^{-\tau} \mathcal{G}_s \left(\frac{s}{L^D} \right)\ ds, \tag{8}$$

where $\mathcal{G}_s(x)$ is constant for $x \ll 1$ and decays rapidly for $x \gg 1$, and τ is the avalanche exponent, $\tau \approx 1.55$.[5] We find

$$P_l(l, L)\ dl = l^{-D} \mathcal{G}_l \left(\frac{l}{L} \right)\ dl, \tag{9}$$

where $\mathcal{G}_l(x)$ is constant for $x \ll 1$ and decays rapidly for $x \to 1$. Note that we have used the scaling relation $\tau = 2 - 1/D$ which is derived from the fact that $\langle s \rangle = L$.

This leads immediately to the probability of having an avalanche of linear size larger than some distance X, $\phi(X) \propto X^{1-D}$. Hence, we may expect to have an avalanche of size $l > X$ within the timescale

$$t = \frac{1}{\phi(X)} = X^{D-1} = X^{\chi}, \tag{10}$$

where we have used the scaling relation $\chi = D - 1$, relating the roughness exponent χ to the avalanche fractal dimension D.[5][2] We use Eq. (10) to obtain an ansatz for $\langle X \rangle(t, L)$, the average linear distance reached by the avalanches in a time t,

$$\langle X \rangle(t, L) = t^{(1/\chi)} g \left(\frac{t}{t_L} \right), \tag{11}$$

where t_L is the timescale after which the avalanches can be expected to have spanned the entire system and $g(x)$ is constant for $x \ll 1$ and proportional to $x^{-(1/\chi)}$ for $x \gg 1$ to ensure that $\langle X \rangle(t, L) \leqslant L$ for all t. By inserting $X = L$ into Eq. (10) we see that $t_L \propto L^{\chi}$, which is consistent with $\langle X \rangle(\infty, L) = L$, as expected.

It is possible, using similar scaling arguments and taking care to use the scaling relation $\tau = 2 - 2/D$ for a bulk driven system[16], to obtain

[2]One might conjecture that the values for the critical exponents are $\tau = (14/9)$, $D = (9/4)$ and $\chi = (5/4)$.

the mean value of the damage for a system in the damaged configuration, $\langle H \rangle \propto L$. This is consistent with our measurement $\langle H \rangle \propto L^{\alpha}$ with $\alpha \approx 1$. Thus, the average damage as a function of time and system size, $\langle H \rangle(t, L)$, may be expressed as

$$\begin{aligned}\langle H \rangle(t, L) &\propto LF_{H\neq 1}(t, L) + \big(1 - F_{H\neq 1}(t, L)\big)\\ &\approx LF_{H\neq 1}(t, L) \quad \text{for } L \gg 1,\end{aligned} \tag{12}$$

where $F_{H\neq 1}(t, L)$ is the fraction of systems in an $H \neq 1$ damaged configuration at time t. If the positions of the perturbed sites are distributed uniformly among the systems in the ensemble, we expect

$$\langle H \rangle(t, L) \propto L \frac{\langle X \rangle(t, L)}{L} = t^{(1/\chi)} G\left(\frac{t}{L^{\chi}}\right), \tag{13}$$

where $G(x) \propto g(x)$ and we have taken $t_0 = 0$. Using the value $\chi = 5/4$, we obtain

$$\langle H \rangle(t, L) = t^{0.80} G\left(\frac{t}{L^{1.25}}\right). \tag{14}$$

A data collapse of the data using these values is shown in Fig. 4. It is in good agreement with Eq. (14), thus supporting our explanation for the appearance of a time dependence in $\langle H \rangle(t, L)$. However, this is only a crude calculation. For instance, there is actually a distribution of times and this will contribute to the time dependence of $\langle H \rangle(t, L)$.

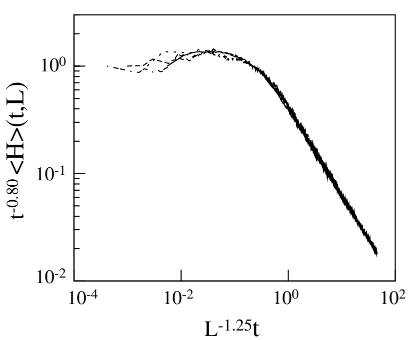

Fig. 4. By plotting $t^{-0.80}\langle H \rangle(t, L)$ versus the rescaled time $x = L^{-1.25}t$ the data for the ensemble average collapses onto a single well defined curve $G(x)$ (see Eq. (14)). This is shown for system sizes $L = 64, 128, 256$ and 512. The number of systems in each ensemble is 10,000.

In conclusion, we have analyzed damage spreading in the Oslo model, showing that damage is unable to evolve in a perturbed system. The damage is statistically time invariant and scale free and thus allows for arbitrarily large values in infinite systems. This phenomenon is due to the Abelian property of the Oslo model, and this generalizes our result to all other models with Abelian properties, including the BTW model,[1] the Manna model[17] and the model for interface depinning in a random medium.[5,9] Thus, many of the classic models of SOC may be considered as lying on the edge of chaos.[14,15] Finally, we have shown how simulations may in fact lead to a time dependent ensemble averaged damage and have calculated this for the case of random placement of the perturbative grain.

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